UDC 531:534.01

EXAMPLES OF APPROXIMATE INVESTIGATION OF THE NEIGHBORHOOD OF STATIONARY MODES OF OSCILLATING MECHANICAL SYSTEMS WITH A SMALL PARAMETER

PMM Vol.43, No. 2, 1979, pp. 235-249

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(Received May 23, 1978)

The investigated mechanical system may be considered to be a set of weakly connected nonlinear oscillators subjected to the action of autonomous forces of various kind (e.g., dissipative forces) and periodic perturbations. Two algorithms are proposed for the derivation of approximate general solution of their equations of motion in the neighborhood of the equilibrium state or periodic motion which are valid over considerable time intervals. Both algorithms are based on the concept of group transformation which affect frequencies.

In algorithm 1 the one-parameter group of symmetry of equations of motion which shifts the small parameter down to zero is used. This reduces the nonlinear input system to a linear system with altered frequencies which is readily integrable. The sought solution is reestablished using Lie's series.

Algorithm 1 is applied in cases in which the total neighborhood of the considered mode is filled by almost-periodic motions, which seldom occurs in the presence of resonances in the system. This algorithm is used here for obtaining nonresonance solutions of the Duffing the Mathieu equations, of equations of the elastic pendulum, and of plane oscillations of a satellite on an elliptic orbit. If in the neighborhood of the investigated stationary mode oscillations are accompanied by an exponential build-up or damping — a characteristic of resonance motions — Algorithm 2 is used.

In the latter case the used symmetry group retains the small parameter so that for each of its fixed value any transformation of the group converts the set of solutions into itself. If the group and the input equations are of the same dimension, the effect of the group is transitive, and all remaining solutions in its neighbrohood are derived from the trivial solution.

The possibility of analyzing intricate situations that occur in the presence of resonances is due to that the determining Lie equations for the group vector field are linear and homogeneous (derivation is carried out in terms of Lie algebra). If these equations in partial derivatives are considered only along the leading motion, they reduce to the well known equations in variations of the theory of motion stability.

It is not possible to determine the characteristic indices of solutions without using equations in variations for the first approximation. To the same extent it is apparently impossible to obtain expansions and indices of exponents in further approximations without resorting to Lie equations which generalize equations in variations.

Algorithm 2 is used here for deriving asymptotic solutions of the Van der Pol, Mathieu, and Duffing equations.

Note that the proposed algorithms are based on a combination of ideas that go back to Linschtedt and Poincaré [1] and of the group method of asymptotic solution derivation. The group aspect of various problems of dynamics was noted in [2, 3] and many other publications (*).

The derivation of solutions valid over considerable time intervals was considered in numerous publications dealing with the method of perturbations [4, 5]. The asymptotic methods reduce the problem to a new system of equations which may not always be integrated. However, if one succeeds to integrate that system, an approximate definition of motion throughout the phase space is obtained.

As opposed to that method, the group-theoretic methods proposed here yield in the described situations simple successively integrable systems which provide, however, a solution of the local problem only, namely the derivation of a general solution in the neighborhood of the investigated resonance.

1. Statement of the problem and description of algorithms. The considered here systems are defined by the equations

$$x_k^{\cdot} = -\omega_k y_k + \varepsilon g_k(x, y, t, \varepsilon) \qquad (1.1)$$
$$y_k^{\cdot} = \omega_k x_k + \varepsilon h_k(x, y, t, \varepsilon), \quad k \leqslant n$$

where the frequencies ω_k are real numbers, ε is a small parameter, and g_k and h_k are analytic functions of their arguments that are 2π -periodic with respect to t or independent of t. Without loss of generality

$$g_k(0, 0, t, \varepsilon) = h_k(0, 0, t, \varepsilon) = 0$$

We have the problem of deriving the general (total) formal solution in the region of stability position in the form of expansion in the small parameter which does not contain secular terms.

It is expedient to write the system of Eqs. (1.1) in complex form. Setting $z_k = x_k + iy_k$ and $f_k = g_k + ih_k$, we obtain

$$z_k^{\cdot} = i\omega_k z_k + \varepsilon f_k(z, \bar{z}, t, \varepsilon), \quad k \leqslant n \qquad (1.2)$$

The conjugate system is not presented since it is not required subsequently: owing to the presence of the small parameter the expressions $i\omega_k z_k$ are the principal terms of the right-hand sides of equations, which reduces the number of equations by half.

Algorithm 1. Let us consider the one-parameter group of transformations which effect the shift of the small parameter $\varepsilon' = \varepsilon + \tau$ (τ is the canonical parameter of the group) and transforms system (1.2) into itself

$$z_k'^{\cdot} = i\omega_k' z_k' + \varepsilon' f_k(z', \bar{z}', t, \varepsilon')$$

The group does not affect time t but transforms frequencies ω_k. If in these transformations we set $\tau = -\varepsilon$, we obtain $\varepsilon' = 0$, and system (1.2) reduces to the

*) Bogoiavlenskii, A. A., Emel'ianova, I. S., Markhashov, L. M., Pavlovskii, Iu. N., and Iakovenko, G. N., The group method of investigation of equations of mechanics of systems with finite number of degrees of freedom. The 3-rd All-Union Chetaev Conference on Motion Stability, Analytical Mechanics, and Motion Control. Theses of Reports. Irkutsk, 1977.

form
$$z_k'^{\bullet} = i\Omega_k z''_k, \quad \Omega_k = \omega_k' \,|_{\tau=-\varepsilon}$$

The condition of invariance of Eqs. (1.2) is conveniently formulated in terms Lie algebra

$$[D_\omega, Z] = 0 \tag{1.3}$$

$$D_\omega = \frac{\partial}{\partial t} + \sum_{k=1}^{n} \left[(i\omega_k z_k + f_k) \frac{\partial}{\partial z_k} + (-i\omega_k \bar{z}_k + f_k) \frac{\partial}{\partial \bar{z}_k} \right]$$

$$Z = \sum_{k=1}^{n} \left[\psi_k \frac{\partial}{\partial z_k} + \bar{\psi}_k \frac{\partial}{\partial \bar{z}_k} + \zeta_{\omega_k} \frac{\partial}{\partial \omega_k} \right] + \frac{\partial}{\partial \varepsilon}$$

where D_ω is the operator of shift along the trajectory of system (1.2), Z is the infinitesimal operator of the group, and $[*, *]$ is the commutator.

The frequency transformation function ζ_{ω_k} is to be selected so that after substitution of ω_k as functions of Ω_k in the expressions for ψ_k the secular terms vanish in the latter. This is most readily obtained by substituting frequencies directly into the equations of motion prior to the formulation of the commutation condition (1.3). Specifically, we shall consider instead of Eqs. (1.2) the following new equations:

$$z_k^{\bullet} = i\chi_k z_k + \varepsilon f_k (z, \bar{z}, t, \varepsilon), \quad k \leqslant n \tag{1.4}$$

$$\chi_k = \Omega_k + \alpha_{k1}\varepsilon + \alpha_{k2}\varepsilon^2 + \ldots \tag{1.5}$$

where Ω_k are new parameters (of frequency), α_{kj} are functions of frequencies selected so that no secular terms appear in expansions at all integration stages; α_{kj} may also depend on variables z_k, $\bar{z}_k$, and t, when we stipulate that the α_{kj} remain constant along the solutions of system (1.4)

$$D\chi_k = 0, \quad D = D_\omega \,|_{\omega_k \to \chi_k} \tag{1.6}$$

where D is the operator of shift along the trajectory of system (1.4).

If we now set

$$\omega_k = \chi_k (\Omega, \varepsilon) \tag{1.7}$$

then by condition (1.6) every solution of system (1.4) with parameters Ω_k will represent some solutions of that system with parameters ω_k, and Ω_k will be defined as implicit functions of ω_k. After frequency transformation and passage to Eqs. (1.4) the group operator assumes the form

$$Y = \sum_{k=1}^{n} \left(\psi_k \frac{\partial}{\partial z_k} + \bar{\psi}_k \frac{\partial}{\partial \bar{z}_k} \right) + \frac{\partial}{\partial \varepsilon} \equiv X + \frac{\partial}{\partial \varepsilon}$$

The condition of commutation is

$$[D, Y] = 0 \quad \text{or} \quad [D, X] = \partial D \,/\, \partial \varepsilon \tag{1.8}$$

which in terms of coordinates yields the determining equations

$$D\psi_k = i\chi_k \psi_k + iz_k X\chi_k + \varepsilon X f_k + iz_k \frac{\partial \chi_k}{\partial \varepsilon} + \frac{\partial}{\partial \varepsilon} (\varepsilon f_k) \tag{1.9}$$

whose solution is sought in the form of series

$$\psi_k = \psi_{k0} + \varepsilon \psi_{k1} + \varepsilon^2 \psi_{k2} + \ldots$$

We represent operators and functions f_k in the form

$$D = D_0 + \sum_{\mu=1}^{\infty} \varepsilon^{\mu} D_{\mu}, \quad X = \sum_{\mu=0}^{\infty} \varepsilon^{\mu} X_{\mu}, \quad f_k = \sum_{\mu=0}^{\infty} \varepsilon^{\mu-1} f_{k\mu} \tag{1.10}$$

$$D_0 = \frac{\partial}{\partial t} + \sum_{k=1}^{n} \left(i\Omega_k z_k \frac{\partial}{\partial z_k} - i\Omega_k \bar{z}_k \frac{\partial}{\partial \bar{z}_k} \right),$$

$$X_{\mu} = \sum_{k=1}^{n} \left(\psi_{k\mu} \frac{\partial}{\partial z} + \bar{\psi}_{k\mu} \frac{\partial}{\partial \bar{z}_k} \right)$$

$$D_{\mu} = \sum_{k=1}^{n} \left[(i\alpha_{k\mu} z_k + f_{k\mu}) \frac{\partial}{\partial z_k} + (-i\alpha_{k\mu} \bar{z}_k + \bar{f}_{k\mu}) \frac{\partial}{\partial \bar{z}_k} \right]$$

Substituting these expressions into Eqs. (1.6) and (1.9) and equating to zero the totality of terms with like powers of ε, we obtain equations

$$D_0 \psi_{ks} = i\Omega_k \psi_{ks} + \sum_{\mu+\nu=s-1} (i\alpha_{k\mu} \psi_{k\nu} + D_{\mu} \psi_{k\nu}) + \sum_{\mu+\nu=s} X_{\mu} f_{k\nu} + \tag{1.11}$$

$$i z_k \left((s+1)\, \alpha_{k,\, s+1} + \sum_{\mu+\nu=s} X_{\mu} \alpha_{k\nu} \right) + (s+1)\, f_{k,\, s+1}$$

$$D_0 \alpha_{ks} = - \sum_{\mu+\nu=s} D_{\mu} \alpha_{k\nu}, \quad s = 0, 1, 2, \ldots \tag{1.12}$$

which are subsequently used for successive determination of functions ψ_{ks}.

Equations (1.11) and (1.12) are successively integrated with the initial conditions $\psi_{ks}(z, \bar{z}, 0) = 0$ (initial conditions may also be selected so as to obtain simpler expressions for functions ψ_{ks}). Functions $\alpha_{k\nu}$ are selected so that functions ψ_{ks} are free of secular terms. After determination of functions ψ_k the final transformations of the group are determined by the Lie series

$$z_k' = z_k + \tau \psi_k + \frac{\tau^2}{2!} Y \psi_k + \frac{\tau^3}{3!} Y^2 \psi_k + \ldots$$

in which it is necessary to set $\tau = -\varepsilon$.

The general solution of system (1.2) is derived by formulas

$$z_k = z_k' \Big|_{z_k = z_k^{\circ} \exp(i\Omega_k t)}$$

where z_k° represent the initial conditions for z_k. Frequencies Ω_k are obtained by inverting formulas (1.5).

The group origin of Ω_k makes possible the proof of reversibility of formulas (1.5) with respect to input frequencies ω_k.

Let the input system be of second order and the frequency transformation function ζ_{ω_k} depend only on ω and ε. Transformation of frequencies is determined by the Lie series by formula

$$\omega' = \omega + \tau \zeta_{\omega}(\omega, \varepsilon) + \frac{\tau^2}{2!} \left(\zeta_{\omega} \frac{\partial \zeta_{\omega}}{\partial \omega} + \frac{\partial \zeta_{\omega}}{\partial \varepsilon} \right) + \ldots \tag{1.13}$$

and the input frequency is determined by its inversion

$$\omega = \omega' - \tau\zeta_\omega(\omega', \varepsilon') + \frac{\tau^2}{2!}\left(\zeta_\omega \frac{\partial \zeta_\omega}{\partial \omega'} + \frac{\partial \zeta_\omega}{\partial \varepsilon'}\right) + \ldots \qquad (1.14)$$

Frequency Ω of the linear system into which the input system is transformed is determined by formula (1.13) with $\tau = -\varepsilon$, and frequency ω by formula (1.14) with $\varepsilon' = 0, \omega' = \Omega$, and $\tau = -\varepsilon$.

Since

$$\zeta_\omega(\Omega, 0) = \alpha_1, \quad \frac{1}{2!}\left(\zeta_\omega \frac{\partial \zeta_\omega}{\partial \omega'} + \frac{\partial \zeta_\omega}{\partial \varepsilon'}\right)\Bigg|_{\varepsilon'=0,\ \omega'=\Omega} = \alpha_2, \ldots$$

function ζ_ω and all its derivatives can be determined at point $(\Omega, 0)$

$$\frac{\partial \zeta_\omega}{\partial \varepsilon'} = 2!\alpha_2 - \zeta_\omega \frac{\partial \zeta_\omega}{\partial \omega'}, \ldots$$

After this the series for function $\zeta_\omega(\omega, \varepsilon)$ and all its derivatives, and consequently also the series for Ω, are derived.

The question of convergence of the obtained expansions is not considered here. It is, however, useful to note that the use of Algorithm 1 for obtaining these series that represent the solution of Eqs. (1.2) are convergent whenever series χ_k for frequencies are convergent. This follows directly from the analyticity of the right-hand sides of Eqs. (1.4) and of initial conditions for ψ_k.

A l g o r i t h m 2. Let the natural frequencies ω_k in system (1.2) be related by l resonance relationships, which implies the existence of $h = n - l$ basic frequencies $\omega_1, \ldots, \omega_h$

$$\omega_{h+\sigma} = \gamma_{\sigma 1}\omega_1 + \ldots + \gamma_{\sigma h}\omega_h, \quad \sigma = 1, \ldots, l$$

where $\gamma_{\sigma j}$ are rational numbers. These relationships can be satisfied either exactly or approximately (to some power of ε).

Natural frequencies can also resonate with the perturbing frequency 1.

As in Algorithm 1, we introduce the series

$$\chi_k = \Omega_k + \sum_{\mu=1}^{\infty} \alpha_{k\mu}\varepsilon^\mu, \quad D\chi_k = 0, \quad k \leqslant n$$

The relationships

$$\Omega_{h+\sigma} = \sum_{j=1}^{h} \gamma_{\sigma j}\Omega_j, \quad \sigma \leqslant l$$

are then exactly satisfied.

If any of the frequencies, for instance ω_1, resonates with unity, we set $\Omega_1 = p/q$, where p and q are integers that correspond to the resonance. When functions $\alpha_{k\mu}$ depend on z, $\bar{z}$, and t, we stipulate the fulfilment of conditions (1.6), i.e. that $D\chi_k = 0$.

We consider here one-parameter groups of system (1.4) that transform it into itself retain parameter ε ($\varepsilon' = \varepsilon$), and shift the trivial solution $z = 0$.

The operators of such groups are of the form

$$Y = \sum_{j=1}^{n}\left(\psi_j \frac{\partial}{\partial z_j} + \overline{\psi}_j \frac{\partial}{\partial \bar{z}_j}\right), \quad Y|_{z=0} \neq 0$$

The commutative relation $[D, Y] = 0$ yields the determining equations

$$D\psi_k = i\chi_k\psi_k + iz_k Y\chi_k + \varepsilon Y f_k$$

that are linear and homogeneous with respect to ψ_j. Hence it is possible to introduce the new unknown functions $\psi_j = \xi_j e^{\Lambda t}$ (Λ is a real quantity and $Y = e^{\Lambda t}X$) in such a way that the exponent $e^{\Lambda t}$ does not appear in the new equations, i. e.

$$D\xi_k = (i\chi_k - \Lambda)\,\xi_k + iz_k X\chi_k + \varepsilon f_k \tag{1.15}$$
$$\Lambda = \beta_1\varepsilon + \beta_2\varepsilon^2 + \ldots$$

The quantities $\beta_1, \beta_2, \ldots$ may depend on z, $\bar{z}$, and t (although frequently they are simply real numbers). In such case the condition

$$D\Lambda = 0 \tag{1.16}$$

is assumed satisfied. In addition, we stipulate that

$$X\Lambda = 0 \tag{1.17}$$

Otherwise the restitution of solution of Eq. (1.2) using operator Y, even when it does not contain secular terms, results in the unavoidable appearance of such terms.

We seek the solution of Eq. (1.15) in the form of series

$$\xi_k = \xi_{k0} + \varepsilon\xi_{k1} + \varepsilon^2\xi_{k2} + \ldots$$

Using expansions (1.10) and equating in Eqs. (1.6) and (1.15) – (1.17) the terms with like powers of ε, we obtain the sequence of equations

$$D_0\xi_{ks} = i\Omega_k\xi_{ks} + \sum_{\mu+\nu=s-1} [(i\alpha_{k\mu} - \beta_\mu)\,\xi_{k\nu} + D_\mu\xi_{k\nu}] + \tag{1.18}$$
$$\sum_{\mu+\nu=s\geqslant 1} X_\mu f_{k\nu} + iz_k \sum_{\mu+\nu=s} X_\mu\alpha_{k\nu}$$
$$D_0\alpha_{ks} = -\sum_{\mu+\nu=s-1} D_\mu\alpha_{k\nu}, \quad s = 0, 1, 2, \ldots$$
$$D_0\beta_s = \sum_{\mu+\nu=s-1} D_\mu\beta_\nu, \quad X_0\beta_s = -\sum_{\mu+\nu=s-1} X_\mu\beta_\nu$$

For the sought functions with index s these equations are readily integrable when functions with indices $j < s$ are already known. The quantities $\alpha_{k\mu}$ and β_μ, and the initial values of functions $\xi_{k\nu}$ are selected on the basis of the condition of elimination in equations of terms that generate secular terms (resonace terms). The solutions of equations for ξ_{k0} are selected in the form $\xi_{k0} = C_k \exp i\Omega_k t$ (C are complex constants). Such functions ξ_{k0} ensure the shift of the trivial solution of Eqs. (1.2).

If n independent solutions of system (1.18) can be found, the sought transitive solution of system (1.15) is obtained in the form of a linear combination of these solutions (with constant coefficients) by virtue of its linearity.

Let $\psi_1, \ldots, \psi_n$ be such solution, i. e.

$$\psi_j = \sum_{k=1}^{n} l_k\psi_{jk}$$

Functions ψ_j define a real transformation group (local) of dimension $2n$ whose final transformations are determined by the Lie series

$$z_j' = z_j + \tau\psi_j + \frac{\tau^2}{2!} Y\psi_j + \frac{\tau^3}{3!} Y^2\psi_j + \ldots \tag{1.19}$$

and ψ_j, $Y\psi_j$, and $Y^2\psi_j$ are, respectively, linear, quadratic, and cubic homogeneous forms of constants $l_1, \ldots, l_n$. Hence the right-hand sides of equalities (1.19) are functions of constants $l_k^* \equiv \tau l_k$ and $z_k' = \Phi_k(z, \bar{z}, t, \varepsilon, l^*)$.

The sought solution of Eqs. (1.2) in the trivial solution neighborhood is obtained using formulas

$$z_k = \Phi_k(0, 0, t, \varepsilon, l^*) \tag{1.20}$$

The solutions obtained using the described algorithms are asymptotic in the following sense. Let z_k be the exact and

$$z_k^{(m)} = z_{k0} + \varepsilon z_{k1} + \varepsilon^2 z_{k2} + \ldots + z_{km}\varepsilon^m$$

the approximate solutions defined by segments of obtained series. Setting $z_k - z_k^{(m-1)} = \varepsilon^m w_k$, $w_k \sim 1$, we obtain for the remainders $z_k - z_k^{(m)}$ differential equations whose right-hand sides are of the $(m+1)$ -st order of smallness. It can be shown using the theorems of existence and uniqueness that functions $|z_k(t) - z_k^{(m)}(t)|$ remain quantities of order ε over times of order ε^{-m}. The same theorem makes it possible to determine also all necessary constants in estimates, including that for initial conditions. Note that the derivation of estimates represents a separate problem.

2. Bounded undamped oscillations. Let us apply Algorithm 1 to Duffing, and Mathieu equations, to equations of motion of an elastic pendulum and equations of plane oscillations of a satellite on an elliptic orbit.

The Duffing equation is of the form

$$x'' + \omega^2 x = \varepsilon c x^3 + \lambda \sin t$$

Using the complex variable $z = x + iy$, $y = -x^{\cdot} / \omega$, we obtain

$$z^{\cdot} = i\omega z - i\varepsilon \frac{c}{8\omega}(z + \bar{z})^3 + \frac{\lambda}{2\omega}(e^{it} - e^{-it}) \tag{2.1}$$

which has a unique solution (see [6]) whose complex form is

$$w = \frac{i\lambda e^{-it}}{2\omega(1+\omega)} + \frac{i\lambda e^{it}}{2\omega(1-\omega)} + \varepsilon\left[\frac{ic\lambda^3}{8\omega(1-\omega^2)^3}\left(\frac{e^{-3it}}{3+\omega} + \frac{e^{3it}}{3-\omega}\right) - \frac{3ic\lambda^3}{8\omega(1-\omega^2)^3}\left(e^{it}\frac{1}{1-\omega} + e^{-it}\frac{1}{1+\omega}\right)\right] + \ldots \tag{2.2}$$

Let us find the first approximation solution of Eq. (2.1) in the neighborhood of the periodic solution (2.1). Setting $z = w + u$

$$\omega = \Omega + \varepsilon\alpha_1 + \ldots \equiv \chi \tag{2.3}$$

we reduce the equation of motion to the form

$$u^{\cdot} = i\Omega u + \varepsilon f_1 + \ldots \tag{2.4}$$

$$f_1 = -\frac{ic}{8\Omega}\left[(u+\bar{u})^3 + \frac{3i\lambda}{1-\Omega^2}(e^{it} - e^{-it})(u+\bar{u})^2 - \frac{3\lambda^2}{(1-\Omega^2)^2}(e^{it} - e^{-it})^2(u+\bar{u})\right]$$

It is now sufficient to use only the first equations of system (1.11)

$$D_0\psi_0 = i\Omega\psi_0 + i\alpha_1 u + f_1(u, \bar{u}, t), \quad D_0\alpha_1 = 0 \tag{2.5}$$

To integrate this, as well as remaining equations of system (1.11) it is convenient to pass to new variables $\gamma = ue^{-i\Omega t}$ and $\bar{\gamma} = \bar{u}e^{i\Omega t}$ which are integrals of the equation $D_0\gamma = 0$. This reduces Eq. (2.5) to the readily integrable form

$$\frac{\partial\psi_0}{\partial t} = i\Omega\psi_0 + i\alpha_1\gamma e^{it} + f_1(\gamma e^{it}, \bar{\gamma}e^{-it}, t), \quad \frac{\partial\alpha_1}{\partial t} = 0 \tag{2.6}$$

To avoid the appearance in the expression for ψ_0 of secular terms, it is necessary to have α_1 of the form

$$\alpha_1 = \frac{3c}{8\Omega}\left(\gamma\bar{\gamma} - \frac{2\lambda^2}{(1-\Omega^2)^2}\right) \equiv \frac{3c}{8\Omega}\left(u\bar{u} - \frac{2\lambda^2}{(1-\Omega^2)^2}\right)$$

Inverting formula (2.3), for the new frequency we obtain the expression

$$\Omega = \omega + \varepsilon\frac{3c}{8\omega}\left(\frac{2\lambda^2}{(1-\omega^2)^2} - u\bar{u}\right) + \ldots$$

and owing to condition $D\chi = 0$ we have $u\bar{u} = u_0\bar{u}_0 + O(\varepsilon)$, where $u_0 = u'(0)$ is the input value of u. Integrating Eq. (2.5) and reverting to variable u, we obtain the transformation formula $u' = u + \varepsilon\psi_0 + \ldots$ whose right-hand side is the general solution of Eq. (2.4) when we set in it $u = u_0e^{i\Omega t}$ (which is equivalent to the condition $\gamma = u_0$). We thus obtain the sought general solution

$$\begin{aligned} u = u_0e^{i\Omega t} + \varepsilon\frac{ic}{8\Omega}\Big[& -\frac{iu_0^3}{2\Omega}e^{3i\Omega t} + \frac{3iu_0\bar{u}_0^2}{2\Omega}e^{-i\Omega t} + \frac{i\bar{u}_0^3}{4\Omega}e^{-3i\Omega t} + \\ & \frac{3\lambda u_0^2}{(1-\Omega^2)(1+\Omega)}e^{-i(2\Omega+1)t} + \frac{6\lambda v_0\bar{u}_0}{(1-\Omega^2)(1-\Omega)}e^{i(2\Omega-1)t} + \\ & \frac{3\lambda\bar{u}_0^2}{(1-\Omega^2)(1-3\Omega)}e^{i(1-2\Omega)t} + \frac{3\lambda\bar{u}_0^2}{(1-\Omega^2)(1+3\Omega)}e^{-i(2\Omega+1)t} + \\ & \frac{6\lambda u_0\bar{u}_0}{(1-\Omega^2)(1-\Omega)}e^{-it} + \frac{6\lambda u_0\bar{u}_0}{(1-\Omega^2)(1+\Omega)}e^{-it} + \frac{3i\lambda^2 u_0}{2(1-\Omega^2)^2}e^{i(\Omega+2)t} - \\ & \frac{3i\lambda^2 u_0}{2(1-\Omega^2)^2}e^{i(\Omega-2)t} + \frac{3i\lambda^2\bar{u}_0}{2(1-\Omega^2)^2(1-\Omega)}e^{-i(\Omega-2)t} - \\ & \frac{3i\lambda\bar{u}_0}{2(1-\Omega^2)^2(1+\Omega)}e^{-i(\Omega+2)t} - \frac{6i\lambda^2\bar{u}_0}{2\Omega(1-\Omega^2)^2}e^{-i\Omega t}\Big] + \ldots \end{aligned}$$

The boundedness of motions (in the first approximation) defined by this formula is consistent with the stability of periodic motion (2.2) (see [6]).

T h e e l a s t i c p e n d u l u m. The equations of motion of such pendulum can be taken in the form [6]

$$q_1'' = -\frac{mg}{cl_1}q_1 + \frac{l_0}{l_1}\left(-q_1q_2 - \frac{1}{2}q_1^3 + q_1q_2^2\right) + O(q^3)$$

$$q_2'' = -q_2 + \frac{l_0}{l_1}\left(-\frac{1}{2}q_1^2 + q_1^2q_2\right) + O(q^3)$$

where m and c are the mass and stiffness of the pendulum, g is the acceleration of gravity, l_0 and l_1 are the lengths of the spring in its free state and in the pendulum equilibrium position, respectively, and q_1 and q_2 are generalized coordinates (linear combinations of Cartesian coordinates). Differentiation is carried out with respect to parameter $\tau = (c/m)^{1/2}t$.

We set $z_k = x_k + iy_k$, $x_1 = q_1$, $y_1 = -x_1 / \omega$, $x_2 = q_2$, and $y_2 = -x_2^{\cdot}$

and write down the equations of motion in the complex form

$$z_1^{\cdot} = i\omega z_1 + \frac{i\gamma}{\omega}\left[\frac{\varepsilon}{4}(z_1+\bar z_1)(z_2+\bar z_2) + \frac{\varepsilon^2}{16}(-2(z_1+\bar z_1)(z_2+\bar z_2)^2 + (z_1+\bar z_1)^3)\right]$$

$$z_2^{\cdot} = iz_2 + i\gamma\left[\frac{\varepsilon}{8}(z_1+\bar z_1)^2 - \frac{\varepsilon^2}{8}(z_1+\bar z_1)(z_2+\bar z_2)\right] + \ldots$$

$$\omega^2 = \frac{mg}{cl_1}, \quad \gamma = \frac{l_0}{l_1}$$

The small parameter is introduced by the substitution $z \to \varepsilon z$. The problem is to determine the motion of pendulum in the neighborhood of its equilibrium position in the second approximation. We use Algorithm 1. Omitting details we adduce the final result

$$z_1 = z_{10}^{\circ} e^{i\Omega_1\tau} - \frac{\gamma}{4\Omega_1\Omega_2} z_{10}^{\circ} z_{20}^{\circ} e^{i(\Omega_1+\Omega_2)\tau} - \frac{\gamma}{4\Omega_1}\frac{\bar z_{10}^{\circ} z_{20}^{\circ}}{\Omega_2 - 2\Omega_1} e^{i(\Omega_2-\Omega_1)\tau} + \frac{\gamma \bar z_{20}^{\circ} z_{10}^{\circ}}{4\Omega_1\Omega_2} e^{i(\Omega_1-\Omega_2)\tau} + \frac{\gamma}{4\Omega_1}\frac{\bar z_{10}^{\circ}\bar z_{20}^{\circ}}{2\Omega_1+\Omega_2} e^{i(\Omega_1+\Omega_2)\tau} + \ldots$$

$$z_2 = z_{20}^{\circ} e^{i\Omega_2\tau} - \frac{i\gamma}{4\Omega_2} z_{10}^{\circ}\bar z_{10}^{\circ} + \frac{i\gamma}{8}\frac{z_{10}^{\circ 2}}{2\Omega_1 - \Omega_2} e^{2i\Omega_1\tau} - \frac{i\gamma}{8}\frac{z_{10}^{\circ 2}}{2\Omega_1+\Omega_2} e^{-2i\Omega_1\tau} + \ldots$$

$$\Omega_1 = \omega + \frac{\gamma}{4\omega}\left[\frac{z_{10}^{\circ}\bar z_{10}^{\circ}}{4}\left(3 - \gamma\frac{3-8\omega^2}{1-4\omega^2}\right) - z_{20}^{\circ}\bar z_{20}^{\circ}\left(1 + \frac{\gamma}{4\omega^2-1}\right)\right]$$

$$\Omega_2 = 1 - \frac{\gamma}{8} z_{10}^{\circ}\bar z_{10}^{\circ}\left(1 + \frac{2\gamma}{4\omega^2-1}\right)$$

where z_{k0}° are arbitrary constants. The formulas show the predominant resonance to be $2\omega = 1$.

T h e M a t h i e u e q u a t i o n. If in the Mathieu equation

$$x^{\cdot\cdot} + \omega^2(1 - h\cos t)x = 0$$

we set $\omega h = 4\varepsilon$ and $y = -x^{\cdot}/\omega$ and introduce the complex variable $z = x + iy$, it assumes the form (2.7)

$$z^{\cdot} = i\omega z + i\varepsilon(z+\bar z)(e^{it} + e^{-it}) \tag{2.7}$$

Let us determine the nonresonance solution in the second approximation of this equation. Using Algorithm 1 we obtain

$$z = z_0 e^{i\Omega t} + \varepsilon\left(z_0 e^{i(\Omega+1)t} - z_0 e^{i(\Omega-1)t} - \frac{\bar z_0}{2\Omega-1} e^{-i(\Omega-1)t} - \frac{z_0}{2\Omega+1} e^{-i(\Omega+1)t}\right) + \varepsilon^2\left[\frac{\Omega z_0 e^{i(\Omega+2)t}}{2\Omega+1} - \frac{\Omega\bar z_0 e^{-i(\Omega+2)t}}{(\Omega+1)(2\Omega+1)} + \frac{\Omega z_0 e^{i(\Omega-2)t}}{2\Omega+1} + \frac{\Omega\bar z_0 e^{-i(\Omega-2)t}}{(2\Omega-1)(\Omega-1)} + \frac{2z_0 e^{-i\Omega t}}{4\Omega^2-1}\right]$$

With an accuracy up to ε^3

$$\Omega = \omega + \frac{4\omega}{1-4\omega^2}\varepsilon^2$$

T h e s a t e l l i t e o n a n e l l i p t i c o r b i t. The equation of plane oscillations of a satellite relative to its center of mass with the latter moving on an elliptical orbit is of the form

$$(1+e\cos v)\frac{d^2\alpha}{dv^2}-2e\frac{d\alpha}{dv}\sin v+\mu\sin\alpha\cos\alpha=2e\sin v \qquad (2.8)$$

where α is the angle of pitch, v is the true anomaly, e is the orbit eccentricity, and $\mu = 3(A-C)/B$ is the ratio of principal moments of inertia of the satellite.

By setting $x = 2\alpha$ we can write the equation of motion in the form of a second order system. The complex form of such system (after expanding $\sin x$ in series)

$$z^{\cdot} = -i\mu^{1/2}a^{-1/2}z-\frac{3a'}{4a}(z-\bar z)+i\,\frac{\mu^{1/2}a^{-1/2}}{48}(z+\bar z)^3+ \qquad (2.9)$$
$$\frac{2}{a^2}-2+\ldots \quad (a=1+e\cos v)$$

According to [5] there exists a 2π-periodic solution of Eq. (2.8) whose generating solution is $\alpha = 0$.

We shall investigate the neighborhood of that periodic solution away from the predominant resonance $2\omega = 1$, on the assumption of smallness of eccentricity e. Using Algorithm 1 we obtain

$$u = u_0e^{i\Omega v}+\varepsilon\left(\frac{2i}{1-\Omega}e^{iv}-\frac{2i}{1+\Omega}e^{-iv}\right)+\varepsilon^2\Big[-u_0\left(\frac{\Omega}{4}+\frac{3}{8}\right)e^{i(\Omega+1)v}+$$
$$u_0\left(\frac{\Omega}{4}-\frac{3}{8}\right)e^{i(\Omega-1)v}+\frac{3\bar u_0}{8(2\Omega+1)}e^{-i(\Omega+1)v}+\frac{3\bar u_0}{8(1-2\Omega)}e^{-i(\Omega-1)v}-$$
$$\frac{u_0^3}{96}e^{3i\Omega v}+\frac{u_0\bar u_0^3}{32}e^{-i\Omega v}+\frac{\bar u_0^3}{192}e^{-3i\Omega v}\Big]+\ldots$$

for $(z=\sqrt{e}\,u\equiv\varepsilon\,u)$.

With an accuracy up to ε^3

$$-\Omega=\sqrt{\mu}-\varepsilon^2\frac{\sqrt{\mu}}{16}u_0\bar u_0 \quad (\sqrt{\mu}\equiv\omega>1)$$

where u_0 is an arbitrary constant.

Thus the complete neighborhood of the investigated periodic motion in the calculated approximation is filled with almost-periodic motions which is in accordance with the fact of stability of periodic motion (*).

3. O s c i l l a t i o n s a c c o m p a n i e d b y e x p o n e n t i a l b u i l d u p o r d a m p i n g [of o s c i l l a t i o n s]. T h e V a n d e r P o h l e q u a t i o n s. Using Algorithm 2 we derive the general solution of the Van der Pohl equation

$$x^{\cdot\cdot}+x=\varepsilon(1-x^2)x^{\cdot}$$

in the neighborhood of equilibrium state $x = x^{\cdot} = 0$. Passing to the complex

*) Sarychev, V. A. and Zlatoustov, V. A. Periodic oscillations of a satellite in the plane of an elliptic orbit. Preprint, Inst. Applied Mathematics, Akad. Nauk SSSR, No. 48, Moscow, 1975.

variable $z = x + iy$, $y = -x^{\cdot}$ and setting $z = \varepsilon u$, we reduce the equation to the form

$$u^{\cdot} = iu + \varepsilon f_1 + \varepsilon^3 f_3 \qquad (3.1)$$
$$f_1 = {}^1/_2 (u - \bar{u}), \quad f_3 = {}^1/_8 (\bar{u}^3 + \bar{u}^2 u - \bar{u} u^2 - u^3)$$

The determining equations (1.18)

$$D_0 \xi_0 = i\Omega \xi_0 \qquad (3.2)$$
$$D_0 \xi_1 = i\Omega \xi_1 - D_1 \xi_0 + (i\alpha_1 - \beta_1)\, \xi_0 + X_0 f_1$$
$$D_0 \xi_2 = i\,\Omega \xi_2 - D_1 \xi_1 - D_2 \xi_0 + (i\alpha_1 - \beta_1)\, \xi_1 + (i\alpha_2 - \beta_2)\, \xi_0 + X_1 f_1$$
$$D_0 \xi_3 = i\Omega \xi_3 - D_1 \xi_2 - D_2 \xi_1 - D_3 \xi_0 + (i\alpha_1 - \beta_1)\, \xi_2 + (i\alpha_2 - \beta_2)\, \xi_1 + (i\alpha_3 - \beta_3)\, \xi_0 + X_0 f_3 + X_2 f_1, \ \ldots$$

(since calculations show that α_3 and β_3 are constants, and their derivatives in Eqs. (3.2) are at once rejected). Integrating the first of Eqs. (3.2) we obtain $\xi_0 = Ce^{i\Omega t}$ (necessarily $\xi_0 |_{u=0} \neq 0$). Resonance terms are eliminated in the right-hand side of the second of Eqs. (3.2) by an appropriate selection of α_1 and β_1. We obtain

$$D_0 \xi_1 = i\Omega \xi_1 - {}^1/_2 \bar{C} e^{-i\Omega t}, \ C = \text{const}\, (\alpha_1 = 0, \ \beta_1 = {}^1/_2)$$

Integration of that equation yields

$$\xi_1 = -\frac{i\bar{C}}{4\Omega} e^{-i\Omega t} + e^{i\Omega t} F_1 (\gamma, \bar{\gamma}) \quad (\gamma = ue^{-i\Omega t}, \ \bar{\gamma} = \bar{u} e^{i\Omega t})$$

where F_1 is an arbitrary function of its arguments.

We reduce the third of Eqs. (3.2) to the form

$$D_0 \xi_2 = i\Omega \xi_2 - \frac{1}{2} (u - \bar{u}) \frac{\partial F_1}{\partial \gamma} - \frac{1}{2} (\bar{u} - u)\, e^{2i\,\Omega t} \frac{\partial F_1}{\partial \bar{\gamma}} +$$
$$(i\alpha_2 - \beta_2)\, Ce^{i\Omega t} - \frac{iC}{8\Omega} e^{i\Omega t} - \frac{1}{2} e^{-i\Omega t} \bar{F}_1$$

in which we eliminate resonance terms by the following selection of α_2 and β_2 :

$$\alpha_2 = {}^1/_8 \Omega, \quad \beta_2 = 0$$

We can set $F_1 = 0$ which yields $D_0 \xi_2 = i\Omega \xi_2$ and, consequently,

$$\xi_2 = e^{i\Omega t} F_2 (\gamma, \bar{\gamma})$$

Functions F_2, α_3, and β_3 are selected so that the resonance terms in the last of Eqs. (3.2) are eliminated

$$F_2 = -{}^1/_8 \bar{C} \gamma^2 - {}^1/_4 C \gamma \bar{\gamma} \equiv -{}^1/_8 \bar{C} u^2 e^{-2i\Omega t} - {}^1/_4 C u \bar{u}$$
$$(\alpha_3 = \beta_3 = 0)$$

Integration of the last of Eqs. (3.2) yields

$$\xi_3 = \frac{i}{2\Omega} \left(\frac{\bar{C}}{32\Omega^2} + \frac{\bar{C}}{4} \gamma \bar{\gamma} + \frac{C}{16} \bar{\gamma}^2 \right) e^{-i\Omega t} + \frac{iC}{4\Omega} \gamma^2 e^{3i\Omega t} +$$
$$\frac{3iC}{32\Omega} \bar{\gamma}^2 e^{-3i\Omega t} + e^{i\Omega t} F_3 (\gamma, \bar{\gamma})$$

Function F_3 whose selection is dictated by the condition of elimination of resonance terms in the equation for ξ_4 (not adduced here) can be of the form

$$F_3 = A_3\gamma^2 + B_3\gamma\bar{\gamma}, \quad A_3 = \frac{3i\bar{C}}{16\Omega}, \quad B_3 = \frac{3iC}{8\Omega}$$

From the same condition we additionally have

$$\alpha_4 = -\frac{1}{128\Omega^3}, \quad \beta_4 = 0$$

Transformation of the group is carried out by formula (1, 19)

$$u' = u + \tau\psi + \frac{\tau^2}{2!}Y\psi + \frac{\tau^3}{3!}Y^2\psi + \ldots \equiv \Phi(z, \bar{z}, t, \varepsilon, l_0, \bar{l}_0)$$

$$\psi = e^{\Lambda t}(\xi_0 + \varepsilon\xi_1 + \varepsilon^2\xi_2 + \varepsilon^3\xi_3 + \ldots), \quad l_0 = \tau C$$

According to Algorithm 2 the general solution of Eq. (3, 1) is calculated by formula

$$u = \Phi(0, 0, \varepsilon, t, l_0, \bar{l}_0)$$

We thus finally obtain

$$u = l_0 e^{(\Lambda+i\Omega)t} - \varepsilon i\frac{\bar{l}_0}{4\Omega}e^{(\Lambda-i\Omega)t} - \varepsilon^2\frac{1}{8}l_0^2\bar{l}_0 e^{(3\Lambda+i\Omega)t} +$$
$$\varepsilon^3\left\{i\frac{\bar{l}_0}{64\Omega^3}e^{(\Lambda-i\Omega)t} + \frac{1}{6}e^{3\Lambda t}\left[\frac{9il_0\bar{l}_0^2}{16\Omega}e^{-i\Omega t} + \frac{9il_0^2\bar{l}_0}{8\Omega}e^{i\Omega t} +\right.\right.$$
$$\left.\left.\frac{3il_0^3}{8\Omega}e^{3i\Omega t} + \frac{3i\bar{l}_0^3}{16\Omega}e^{-3i\Omega t}\right]\right\} + \ldots$$

where l_0 is an arbitrary constant.

With an accuracy to ε^4 we have $\Lambda = {}^1/_2\,\varepsilon$. Inverting the formula for frequency

$$1 = \Omega + \frac{\varepsilon^2}{8\Omega} - \frac{\varepsilon^4}{128\Omega^3} + \ldots$$

we obtain

$$\Omega = 1 - \frac{1}{8}\varepsilon^2 - \frac{1}{128}\varepsilon^4 + \ldots$$

T h e M a t h i e u e q u a t i o n. Analysis of the neighborhood of the 4π-periodic resonance solution of the Mathieu equation (2, 7) yields the general solution

$$z_1 = e^{\Lambda_1 t}\,\Phi^*\,(z_0 = e^{i\theta}, \quad \alpha_1 = -ie^{-2i\theta}), \quad z_2 = e^{\Lambda_2 t}\,\Phi^*\,(z_0 = e^{-i\theta},$$
$$\alpha_1 = -ie^{2i\theta})\,|$$
$$\Phi^* = z_0 e^{{}^1/_2 t} + \varepsilon\,(z_0 e^{{}^3/_2 it} - z_0 e^{-{}^1/_2 it} - {}^1/_2\bar{z}_0 e^{-{}^3/_2 it}) +$$
$$\varepsilon^2\,[-{}^1/_6\bar{z}_0 e^{-{}^5/_2 it} - ({}^1/_2\alpha_1\bar{z}_0 + iz_0)e^{-{}^3/_2 it} + {}^1/_2 i\bar{z}_0 e^{-{}^1/_2 it} -$$
$$2i\bar{z}_0 e^{{}^3/_2 it} + {}^1/_2 iz_0 e^{{}^5/_2 it}]$$

where l_1 and l_2 are arbitrary constants and θ is a parameter.

The natural frequency and the index are determined (with accuracy to ε^3) by formulas

$$\omega = {}^1/_2 - \varepsilon\cos 2\theta + {}^1/_2\,\varepsilon^2, \quad \Lambda_1 = \varepsilon\sin 2\theta, \quad \Lambda_2 = -\varepsilon\sin 2\theta$$

For each fixed ω we obtain completely determined value of θ, indices Λ_1 and Λ_2 that correspond to that value, and the solution z.

The Duffing equation. When the natural frequency of oscillation is close to that of perturbations $\omega \approx 1$, the Duffing equation assumes the form (see [6])

$$x^{\cdot\cdot} + x^{\cdot} = \varepsilon a x + \varepsilon c x^3 + \varepsilon \lambda' \sin t$$

In the complex form

$$z^{\cdot} = iz - i\frac{\varepsilon a}{2}(z + \bar{z}) - i\,\frac{\varepsilon c}{8}(z + \bar{z})^3 + \frac{\varepsilon \lambda'}{2}(e^{-it} - e^{it}) \qquad (3.3)$$

The 2π-periodic solutions of this equation are determined by formula (see [6])

$$w = w_0 + \varepsilon w_1 + \varepsilon^2 w_2 + \ldots, \quad w_0 = -i\rho e^{it} \qquad (3.4)$$

$$w_1 = -\frac{ic\rho^3}{16}e^{3it} - \frac{ic\rho^3}{32}e^{-3it} - iN_1 e^{it}, \quad N_1\left(a + \frac{9}{4}c\rho^2\right) = \frac{3c^2\rho^5}{128}$$

The amplitude ρ is determined by the set of positive solutions of equations

$$a\rho + {}^3/_4\, c\rho^3 \pm \lambda' = 0 \ (\lambda' > 0) \qquad (3.5)$$

There can be one or three of such solutions. The case of multiple roots of Eqs. (3.4) is not considered here. Using Algorithm 2 we obtain the general solution of Eq. (3.3) in the neighborhood of periodic motions. The substitution $z = w + \varepsilon u$ reduces this equation with allowance for expansion (3.4)) to the form

$$u^{\cdot} = iu + \varepsilon f_1 + \varepsilon^2 f_2 + \ldots$$

$$f_1 = (u + \bar{u})\left[\frac{3ic\rho^2}{8}(e^{2it} + e^{-2it}) - \frac{i}{2}\left(a + \frac{3c\rho^2}{2}\right)\right]$$

$$f_2 = i(u + \bar{u})\left[-\frac{3c\rho}{2}N_1 + \frac{3c\rho}{4}\left(N_1 - \frac{c\rho^3}{32}\right)(e^{2it} + e^{-2it}) + \frac{3c^2\rho^4}{128}(e^{4it} + e^{-4it})\right] + \frac{3c\rho}{8}(u + \bar{u})^2(e^{-it} - e^{it})$$

The determining equation for ξ_0 is $D_0\xi_0 = i\xi_0$ (see (1.18)). We assume its solution to be of the form $\xi_0 = Ce^{it}$ (C = const).

Function ξ_1 is obtained by integrating the equation

$$D_0\xi_1 = i\xi_1 - D_1\xi_0 + a_1\xi_0 + X_0 f_1 \quad (a_k \equiv i\alpha_k - \beta_k)$$

After substitution of specific expressions for ξ_0 and f_1 we have

$$D_0\xi_1 = i\xi_1 + a_1 Ce^{it} + C\frac{3ic\rho^2}{8}(e^{3it} + e^{-it}) - C\frac{i}{2}\left(a + \frac{3c\rho^2}{2}\right)e^{it} + \bar{C}\frac{3ic\rho^2}{8}(e^{it} + e^{-3it}) - \bar{C}\frac{i}{2}\left(a + \frac{3c\rho^2}{2}\right)e^{-it}$$

Equating to zero the totality of resonance terms in the right-hand side, we obtain

$$a_1 C - C\frac{i}{2}\left(a + \frac{3c\rho^2}{2}\right) - \frac{3ic\rho^2}{8}\bar{C} = 0 \quad (C = re^{i\theta})$$

from which

$$a_1 \equiv i\alpha_1 - \beta_1 = -\frac{3c\rho^2}{8}\sin 2\theta + i\left(\frac{a}{2} + \frac{3c\rho^2}{4} - \frac{3c\rho^2}{8}\cos 2\theta\right)$$

Since $\omega = 1 + \alpha_1\varepsilon + \ldots$, hence at exact resonance $(\omega = 1)$

$$\alpha_1 \equiv \frac{a}{2} + \frac{3c\rho^2}{4} - \frac{3c\rho^2}{8}\cos 2\theta = 0 \tag{3.6}$$

Further we have

$$\beta_1 = \frac{3c\rho^2}{8}\sin 2\theta \tag{3.7}$$

It can be readily verified by comparing conditions (3.6) and (3.5) that it is possible to satisfy the equation by a real value of parameter θ only when the amplitude equations (3.5) have all their three roots $\rho_1 < \rho_2 < \rho_3$ real and $\rho = \rho_2$.

Thus in this case we have to deal with the neighborhood of periodic solution that corresponds to the middle root $\rho = \rho_2$.

Evidently the two values of β_1

$$\beta_I^+ = \frac{3c\rho^2}{8}\sin 2\theta, \quad \beta_I^- = -\frac{3c\rho^2}{8}\sin 2\theta$$

and, consequently, the two values of the index

$$\Lambda_1 = \varepsilon\frac{3c\rho^2}{8}\sin 2\theta + \ldots, \quad \Lambda_2 = -\varepsilon\frac{3c\rho^2}{8}\sin 2\theta + \ldots$$

correspond to $\cos 2\theta$ determined by formula (3.6).

Taking into consideration that subsequently the exponents with these indices appear in the solution of Eq. (3.3), we can see that the related periodic solution is unstable. This agrees with the statements in [6].

Further

$$\xi_1 = \frac{3c\rho^2}{8}\left(\frac{C}{2}e^{3it} - \frac{C}{4}e^{-3it}\right) + \frac{1}{2}\left(\bar{C}\frac{3c\rho^2}{4} - C\frac{3c\rho^2}{8} + \bar{C}\frac{a}{2}\right)e^{-it} +$$
$$e^{it}F_I(\gamma, \bar{\gamma})$$

We select α_2, β_2, and function F_1 so that the resonance terms in the equation for ξ_2

$$D_0\xi_2 = i\xi_2 - D_1\xi_1 - D_2\xi_0 + a_1\xi_1 + a_2\xi_0 + X_0f_2 + X_1f_1$$

are eliminated

After necessary transformations we obtain

$$r = \rho, \quad F_1 = A\gamma + B\bar{\gamma}, \quad A = i\frac{\cos 2\theta}{\cos^3\theta}, \quad B = -\frac{\sin\theta}{\cos^3\theta}e^{i\theta}$$

These formulas show that ξ_1 has a singularity at $\cos\theta = 0$ to which, as can be readily see, corresponds the case of multiple roots of the amplitude equation (3.5), which was eliminated from our investigation. The following two solutions of determining equations:

$$\xi_0^+ = \rho e^{i\theta}e^{it}, \quad \xi_0^- = \rho e^{-i\theta}e^{it}$$
$$\xi_1^+ = \frac{3c\rho^3}{8}\left(e^{i\theta}\frac{e^{3it}}{2} - e^{-i\theta}\frac{e^{-3it}}{4}\right) + \frac{1}{2}\left(e^{-i\theta}\frac{3c\rho^3}{4} - e^{i\theta}\frac{3c\rho^3}{8} +\right.$$
$$\left.e^{-i\theta}\frac{\rho a}{2}\right)e^{-it} + A^+u + B^+\bar{u}e^{2it}, \quad A^+ = A, \quad B^+ = B$$

$$\xi_1^- = \frac{3c\rho^3}{8}\left(e^{-i\theta}\frac{e^{3it}}{2} - e^{i\theta}\frac{e^{-3it}}{4}\right) + \frac{1}{2}\left(e^{i\theta}\frac{3c\rho^3}{4} - e^{-i\theta}\frac{3c\rho^3}{8} + e^{i\theta}\frac{\rho a}{2}\right)e^{-it} + A^-u + B^-\bar{u}e^{2it}, \quad A^- = A, \ B^- = \frac{\sin\theta}{\cos^3\theta}e^{-i\theta}$$

correspond to θ and $-\theta$ that satisfy condition (3.6).

Reverting now to functions $\psi = \xi e^{\Lambda t}$ we obtain a two-parameter group which is determined by the set of operators $Y = \psi\,\partial/\partial u + \bar{\psi}\,\partial/\partial u$ where in the first approximation

$$\psi = l_1(\xi_0^+ + \varepsilon\xi_1^+)e^{\Lambda_1 t} + l_2(\xi_0^- + \varepsilon\xi_1^-)e^{\Lambda_2 t} \equiv \psi(l_1, l_2, u, \bar{u}, t, \varepsilon)$$

l_1 and l_2 are arbitrary constants. The first approximation of the sought general solution is determined by formula

$$u = \psi(l_1^*, l_2^*, 0, 0, t, \varepsilon) + \frac{1}{2}\left(\psi(l_1^*, l_2^*, 0, 0, t, \varepsilon)\frac{\partial\psi}{\partial u}\Big|_{u=0} + \bar{\psi}(l_1^*, l_2^*, 0, 0, t, \varepsilon)\frac{\partial\psi}{\partial\bar{u}}\Big|_{u=0}\right)$$

where $l_1^* = \tau l_1$ and $l_2^* = \tau l_2$ are arbitrary constants.

Let us now turn to the neighborhood of other periodic solutions of the Duffing equation. The nonresonance Duffing equation (2.1) is converted into the resonance equation (3.3) by the substitution $\omega^2 - 1 = a\varepsilon$ and $\lambda' = \varepsilon\lambda$. Obviously the related periodic solutions of Eqs. (2.1) and (3.3) convert into one another. Simple reasoning and calculations show that the transformation is achieved for solutions with $\rho = \rho_1$ and $\rho = \rho_2$.

The corresponding neighborhoods of these solutions transform into one another, which makes possible their determination using the indicated substitution.

The author thanks V. M. Alekseev for important critical remarks.

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Translated by J. J. D.